CONTACT PROBLEMS FOR AN ELASTIC SPHERE

PMM Vol. 35, №1, 1971, pp. 61-70
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(Received April 21, 1970)

It is shown that in the general case the contact pressure has a square root singularity at the interface of the boundary conditions. This singularity is extracted from the solution. A Fredholm equation of the second kind is obtained for the bounded additive part of the contact pressure, which is solved by an asymptotic method. As an illustration, the problem of a heavy sphere with a spherical rigid foundation of similar radius is solved numerically. A comparison is made with the Hertz solution.

Hertz [1, 2] posed and solved the problem of the contact of elastic solids under the assumption that the contacting bodies can be replaced by elastic half-spaces for small domains of contact.

The contact problem for a sphere with a given interface of the boundary conditions (a stamp with angular points) is reduced in [3, 4] to the determination of certain coefficients in the dual series-equations containing Legendre polynomials; a method is indicated which permits reduction of the solution of the obtained dual series-equations to the solution of infinite systems of linear equations.

Contact problems (including the problem with a previously unknown interface of the boundary conditions) are investigated below on the basis of the closed solution of the first boundary value problem for a sphere obtained in [5].

It can be shown [5] that the axisymmetrical loading normal to a sphere $r = R$

$$\sigma_r = N(\theta) \qquad \text{for } r = R,\ 0 < \theta < \pi$$

produces the following radial displacements on this sphere

$$u(\theta) = \frac{R}{2\pi G}\int_0^\pi N(\alpha)\, H(\theta, \alpha) \sin\alpha\, d\alpha \tag{1}$$

$$H(\theta, \alpha) = \frac{\pi}{2}\frac{1-2\nu}{1+\nu} + 4(1-\nu)\,U(1) + \operatorname{Re}\int_0^1 \left(\frac{A}{y^\lambda} + \frac{1}{y^2}\right) U(y)\, dy \tag{2}$$

The function U in the kernel of (2) is expressed in terms of complete elliptic integrals of the first kind $K(k)$ and is the following:

$$U(y) = U(y, \theta, \alpha) = \frac{K(k)}{h} - \frac{\pi}{2}(1 + y\cos\theta\cos\alpha) \tag{3}$$

$$h^2 = (1-y)^2 + 4y\sin^2\frac{\theta+\alpha}{2}, \qquad k^2h^2 = 4y\sin\theta\sin\alpha$$

The constants A and λ in (2) depend only on the Poisson's ratio ν and are given in the form

$$A = 8\nu^2 - 8\nu + 1 + i\,\frac{16\nu^3 - 16\nu^2 - 4\nu + 5}{\sqrt{3-4\nu^2}}, \qquad \lambda = \frac{1-2\nu}{2} + i\,\frac{\sqrt{3-4\nu^2}}{2} \tag{4}$$

Let us consider the problem of impression of a sphere $r \leqslant R$ into a rigid axisymmetric foundation given in the spherical (r, θ, φ) coordinate system by the following equation (Fig. 1):

$$r = R\,[1 + \rho(\theta)], \qquad \rho(0) = 0 \tag{5}$$

It is assumed that the sphere is free of shear stress resultants and is deformed under the effect of the normal loading

$$N(\theta) = \begin{cases} Q(\theta) & \text{for } \gamma < \theta < \pi \\ \sigma(\theta) & \text{for } 0 < \theta < \gamma \end{cases} \tag{6}$$

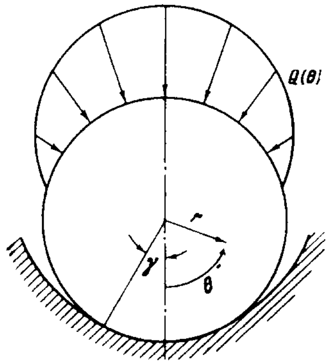

Fig. 1

where $Q(\theta)$ is given and $\sigma(\theta)$ is the required contact pressure.

The circle $\theta = \gamma$ on the sphere $r = R$ bounds the domain of contact. The problem is solvable under the condition

$$\int_0^{\gamma} \sigma(\alpha) \sin\alpha \cos\alpha \, d\alpha = -\frac{Z}{2\pi R^2} \tag{7}$$

where Z is the equivalent external pressure $Q(\theta)$.

The condition of contact between a sphere and the stamp (5) can be written in the form

$$u(\theta) = R\,[-a\cos\theta + \rho(\theta)], \qquad 0 < \theta < \gamma \tag{8}$$

where a is the approach beteen the stamp and the center of the sphere.

Substitution of the boundary conditions (6) and (8) into (1) results in an integral equation in the contact pressure

$$\frac{R}{2\pi G} \int_0^{\gamma} \sigma(\alpha) H(\theta, \alpha) \sin\alpha \, d\alpha = v(\theta), \qquad 0 < \theta < \gamma \tag{9}$$

Here $v(\theta)$ denotes the following function given to the accuracy of a

$$v(\theta) = R\left[-a\cos\theta + \rho(\theta) - \frac{1}{2\pi G}\int_{\gamma}^{\pi} Q(\alpha) H(\theta, \alpha) \sin\alpha \, d\alpha\right] \tag{10}$$

Let us convert (9) into a Fredholm integral equation of the second kind. Let us make the change of variables $\operatorname{tg} {}^1\!/_2\theta = \varepsilon x$, $\operatorname{tg} {}^1\!/_2\alpha = \varepsilon t$, $\varepsilon = \operatorname{tg} {}^1\!/_2 \gamma$

Moreover, let us introduce the notation

$$q(x) = \frac{4\varepsilon^2 \sigma^\circ(x)}{(1 + \varepsilon^2 x^2)^{3/2} G}, \quad w(x) = \frac{2 v^\circ(x)}{(1 + \varepsilon^2 x^2)^{1/2} R}, \quad \theta_1 = \frac{1 - \nu}{2\pi}$$

$$S(x, t) = \frac{t}{\sqrt{(1 + \varepsilon^2 x^2)(1 + \varepsilon^2 t^2)}} \left[\frac{1}{2}\,\frac{1 - 2\nu}{1 + \nu} - 2(1 - \nu) \times \right.$$
$$\left. \times \left(1 + \frac{1 - \varepsilon^2 x^2}{1 + \varepsilon^2 x^2}\,\frac{1 - \varepsilon^2 t^2}{1 + \varepsilon^2 t^2}\right) + \frac{1}{\pi} \operatorname{Re} \int_0^1 \left(\frac{A}{y^{\lambda}} + \frac{1}{y^2}\right) U^\circ(y)\, dy\right] \tag{11}$$

Here

$$\sigma^\circ(x) = \sigma(2 \operatorname{arc\,tg} \varepsilon x), \qquad v^\circ(x) = v(2 \operatorname{arc\,tg} \varepsilon x)$$
$$U^\circ(y) = U^\circ(y, x, t) = U(y, 2 \operatorname{arc\,tg} \varepsilon x, 2 \operatorname{arc\,tg} \varepsilon t)$$

After the manipulations mentioned, (9) becomes

$$\int_0^1 q(t)\left[\frac{4t}{x + t} K\left(\frac{2\sqrt{xt}}{x + t}\right) + \frac{\varepsilon}{\theta_1} S(x, t)\right] dt = \frac{\varepsilon}{\theta_1} w(x), \qquad 0 < x < 1 \tag{12}$$

The equilibrium condition (7) is hence rewritten as

$$\int_0^1 q(t)\,\frac{(1-\varepsilon^2t^2)\,t}{(1+\varepsilon^2t^2)^{3/2}}\,dt=-\frac{Z}{2\pi R^2G} \tag{13}$$

Let us note that the axisymmetric contact problem for an elastic half-space results in the integral equation

$$\int_0^1 q(t)\,\frac{4t}{x+t}\,K\left(\frac{2\sqrt{xt}}{x+t}\right)dt=f(x),\qquad 0<x<1 \tag{14}$$

The solution of this equation in quadratures is given in [6] as

$$q(x)=\frac{c}{\sqrt{1-x^2}}-\frac{\sqrt{1-x^2}}{\pi^2}\int_0^{\pi/2}d\psi\int_0^{\pi/2}\Delta f\left(\sqrt{1-(1-x^2)\sin^2\psi}\,\sin\alpha\right)\times$$

$$\times\sin\psi\sin\alpha\,d\alpha \tag{15}$$

where

$$\Delta f(t)=\frac{1}{t}f'(t)+f''(t),\qquad c=\frac{1}{\pi^2}\left[f(0)+\int_0^1\frac{f'(t)\,dt}{\sqrt{1-t^2}}\right] \tag{16}$$

It can be shown that (15) admits of an equivalent representation

$$q(x)=\frac{c}{\sqrt{1-x^2}}-\frac{1}{\pi^2}\int_0^1\Delta f(t)\,L(x,t)\,dt \tag{17}$$

$$L(x,t)=\frac{t}{x+t}\,K\left(\frac{2\sqrt{xt}}{x+t}\right)-F\left(\frac{x}{t},\,t\right) \tag{18}$$

Let us note the following property of the incomplete elliptic integral of the first kind:

$$F(k,x)=\int_0^x\frac{dt}{\sqrt{(1-t^2)(1-k^2t^2)}}=\frac{1}{k}F\left(\frac{1}{k},\,kx\right) \tag{19}$$

It aids in establishing that

$$xL(x,t)=tL(t,x) \tag{20}$$

Let us continue the investigation of the integral equation of the first kind (12) by the method of regularization. Let us consider the integral in the right side of (17) as an operator, and let us act on it on the left with (12). We then obtain an equation equivalent to (12)

$$q(x)-\frac{\varepsilon}{\theta_1\pi^2}\int_0^1 q(t)\,B(x,t)\,dt=\frac{c}{\sqrt{1-x^2}}-\frac{\varepsilon}{\theta_1\pi^2}\int_0^1\Delta w(t)\,L(x,t)\,dt$$

$$0<x<1 \tag{21}$$

where

$$B(x,t)=\int_0^1\left[\frac{1}{z}S_z'(z,t)+S_{zz}''(z,t)\right]L(x,z)\,dz \tag{22}$$

Let us introduce a function $p(x)$ into the considerations such that

$$q(x)=\frac{c}{\sqrt{1-x^2}}+\frac{\varepsilon}{\theta_1\pi^2}\,p(x) \tag{23}$$

It is the solution of a Fredholm integral equation of the second kind

$$p(x) - \frac{\varepsilon}{\theta_1\pi^2}\int_0^1 p(t)\,B(x, t)\,dt = g(x), \quad 0 < x < 1 \tag{24}$$

$$g(x) = -\int_0^1 \Delta w(t)\,L(x, t)\,dt + c\int_0^1 \frac{B(x, t)}{\sqrt{1-t^2}}\,dt \tag{25}$$

Condition (13), which can be written according to (23) as

$$c\left[\frac{2}{1+\varepsilon^2} - \frac{\operatorname{arctg}\varepsilon}{\varepsilon}\right] + \frac{\varepsilon}{\theta_1\pi^2}\int_0^1 p(t)\,\frac{(1-\varepsilon^2 t^2)\,t}{(1+\varepsilon^2 t^2)^{3/2}}\,dt = -\frac{Z}{2\pi R^2 G} \tag{26}$$

is used to determine c.

Substituting (23) into (12) and taking into account that

$$w(0) = -2a - \frac{1}{\pi G}\int_\gamma^\pi Q(\alpha)\,H(0, \alpha)\sin\alpha\,d\alpha \tag{27}$$

we obtain the connection between the approach a and the factor c

$$w(0) = c\left[\frac{\theta_1\pi^2}{\varepsilon} + \int_0^1 \frac{S(0, t)}{\sqrt{1-t^2}}\,dt\right] + \frac{1}{\pi^2}\int_0^1 p(t)\left[2\pi + \frac{\varepsilon}{\theta_1}S(0, t)\right]dt \tag{28}$$

It can be shown that the kernel $B(x, t)$ is bounded for $t \neq x$ and has a logarithmic singularity at $t = x$. As is customary, let us assume that $w(x) \in C^2$ in the segment [0, 1]. Then $g(x) \in C$ in [0, 1].

The kernel $B(x, t)$ is square integrable in the set of variables x and t.

$$\int_0^1\int_0^1 |B(x, t)|^2\,dx\,dt = b^2 < \infty$$

and moreover

$$\int_0^1 |B(x, t)|^2\,dt \leqslant T < \infty$$

Furthermore, the kernel $B(x, t)$ is continuous in x in the large [7] in the segment [0, 1].

According to the properties listed for the integral equation (24), its solution $p(x) \in C$ in [0, 1] and because of the theorem of I. Shur, for $\varepsilon \leqslant \theta_1\pi^2 b$ it can be represented by a uniformly convergent Neumann power series in ε, where the series converges no more slowly than a progression with the denominator $\varepsilon b / \theta_1\pi^2$ according to [7].

Let us turn to the solution of the integral equation (24).

It can be shown that the function $S(x, t)$ in (11) is representable by the following convergent series for $\varepsilon < 1$, $0 \leqslant x, t \leqslant 1$:

$$S(x, t) = t\left[\sum_{n=0}^{\infty} f_n(x, t)\,\varepsilon^n + \ln\varepsilon\sum_{n=0}^{\infty} g_{2n}(x, t)\,\varepsilon^{2n}\right] \tag{29}$$

where $f_n(x, t)$ and $g_{2n}(x, t)$ are continuous functions with an integrable second derivative. In particular

$$f_0(x, t) = -\varkappa\ln\max(x, t) + c_0, \quad f_1(x, t) = 4c_1(x + t)\,E\left(\frac{2\sqrt{xt}}{x+t}\right)$$

$$f_2(x, t) = {}^1/_4 c_2(x^2 + t^2)\ln\max(x, t) + ({}^1/_2\varkappa - {}^1/_4 c_2)\max(x^2, t^2) +$$

$$+ {}^1/_4 c_3 (x^2 + t^2)$$

$$g_0 (x, t) = -\varkappa, \qquad g_2 (x, t) = {}^1/_4 c_2 (x^2 + t^2)$$

Here $E(k)$ is the complete elliptic integral of the second kind, and the coefficients c_i are expressed in terms of the constants A and λ in (4), as follows:

$$c_0 = \frac{1}{2}\frac{1-2\nu}{1+\nu} - 4(1-\nu) + \frac{1}{2}\operatorname{Re} A \int_0^1 \frac{y^{2-\lambda}-1}{1-y}\,dy$$

$$c_1 = {}^1/_2\pi\,[\operatorname{Re} A\,(2-\lambda) - 3\varkappa], \qquad c_2 = 10\varkappa - 2\operatorname{Re} A\,(2-\lambda)(1+\lambda)$$

$$c_3 = -2c_0 + 16(1-\nu) + 20\varkappa - 4\operatorname{Re} A\,(2-\lambda) -$$

$$-4\operatorname{Re} A \int_0^1 \frac{y^{2-\lambda} - 1 - (2-\lambda)(y-1) - {}^1/_2(2-\lambda)(1-\lambda)(y-1)^2}{y}\left[\frac{1}{(1-y)^3} - 1\right]dy$$

where

$$\varkappa = (1-2\nu)^{\,2}$$

Substituting the series (29) into (22), we obtain an analogous expansion for the kernel of the integral equation

$$B(x, t) = \sum_{n=0}^{\infty} F_n(x, t)\,\varepsilon^n + \ln\varepsilon \sum_{n=1}^{\infty} G_{2n}(x, t)\,\varepsilon^{2n} \qquad (30)$$

where

$$F_0(x, t) = -\varkappa\,L(x, t), \qquad G_2(x, t) = c_2 t\sqrt{1-x^2}$$

$$F_1(x, t) = c_1 \int_0^1 L(x, s)\frac{4t}{s+t} K\left(\frac{2\sqrt{st}}{s+t}\right) ds$$

$$F_2(x, t) = t\left\{c_3\sqrt{1-x^2} + c_2\int_0^1 \ln\max(s, t)\,L(x, s)\,ds + \right.$$

$$\left. + \varkappa\left[tL(x, t) + 2\int_0^1 h(s-t)\,L(x, s)\,ds\right]\right\} \qquad (31)$$

Here $h(x)$ is the Heaviside function. In order to be sequential, let us also expand the right side $g(x)$ of the integral equation (24) in a series of the type (29).

As is usual, let us assume that the right side of the initial equation (12) is representable in the neighborhood of the point $x = 0$ by the series

$$w(x) = -a\left[2 + \sum_{n=1}^{\infty}(-1)^n \frac{1\cdot 3 \ldots (2n-1)(1+4n)}{2^{n-1}\,n!}(\ x)^{2n}\right] + \sum_{n=0}^{\infty} A_{2n}(\varepsilon x)^{2n} \qquad (32)$$

which is uniformly convergent for $\varepsilon < 1$.

The coefficients A_{2n} are determined by the given external loading $Q(\theta)$ and the shape of the stamp, and are independent of the approach a.

Substituting (3.2) into the first integral in (25), we obtain

$$-\int_0^1 \Delta w(t)\,L(x, t)\,dt = \sqrt{1-x^2}\sum_{n=1}^{\infty}\left[a(-1)^n \frac{1\cdot 3\ldots(2n-1)(1+4n)}{2^{n-1}\,n!} - A_{2n}\right] \times$$

$$\times\, \varepsilon^{2n}\frac{2n+1}{2n-1}\beta_n E_{n-1}(x^2), \qquad \beta_n = \frac{n\sqrt{\pi}\,\Gamma(n+1)}{\Gamma(n+{}^3/_2)} \quad \left(n > \frac{1}{2}\right) \qquad (33)$$

where the $E_n(x)$ are Jacobi polynomials.

The formula

$$\int_0^1 t^{2n-2} L(x,t)\,dt = \frac{2n+1}{2n-1}\frac{\beta_n}{(2n)^2} E_{n-1}(x^2)\sqrt{1-x^2}$$

utilized in deriving (33), was obtained after some manipulation and a comparison with the results in [8, 9].

Substituting its corresponding series (30) in place of $B(x, t)$ into the second integral of (25), we obtain

$$\int_0^1 \frac{B(x,t)}{\sqrt{1-t^2}}\,dt = \sum_{n=0}^{\infty} M_n(x)\varepsilon^n + \ln\varepsilon \sum_{n=1}^{\infty} N_{2n}(x)\varepsilon^{2n} \tag{34}$$

$$M_n(x) = \int_0^1 \frac{F_n(x,t)}{\sqrt{1-t^2}}\,dt, \qquad N_{2n}(x) = \int_0^1 \frac{G_{2n}(x,t)}{\sqrt{1-t^2}}\,dt$$

In particular

$$M_0(x) = -\varkappa J_1(x), \qquad M_1(x) = c_1\pi^2\sqrt{1-x^2}$$

$$M_2(x) = (c_3 + 2\varkappa)\sqrt{1-x^2} + \varkappa\,[J_1(x) - 3J_3(x)] + c_2\,[J_2(x) - J_3(x)]$$

$$N_2(x) = c_2\sqrt{1-x^2}$$

Here

$$J_1(x) = \int_0^1 \frac{L(x,t)}{\sqrt{1-t^2}}\,dt, \qquad J_2(x) = \int_0^1 \ln\left(1+\sqrt{1-t^2}\right) L(x,t)\,dt \tag{35}$$

$$J_3(x) = \int_0^1 \sqrt{1-t^2}\, L(x,t)\,dt, \qquad J_4(x) = \int_0^1 (1-t^2)^{3/2} L(x,t)\,dt$$

An asymptotic representation of these integrals as $x \to 1$ is easily obtained

$$J_1(x) \sim -\sqrt{1-x^2}\ln\sqrt{1-x^2}, \quad J_i \sim k_i\sqrt{1-x^2}, \quad k_i = \text{const}, \quad i = 2, 3, 4$$

The integrals (35) reach a maximum at $x = 0$

$$J_1(0) = {}^1/_8\pi^2, \qquad J_2(0) = {}^1/_8\pi^2 - 2 + \ln 4$$

$$J_3(0) = {}^1/_{16}\pi^2 + {}^1/_4, \quad J_4(0) = {}^3/_{64}\pi^2 + {}^1/_4$$

To solve the integral equation (24), let us substitute the series (30), (33), (34) therein and let us apply the asymptotic method of Vorovich and Aleksandrov [10]. The form of the solution is at once established.

Let us note that the constants c and a are linear in the right side of the integral equation (24). Because of the linearity of the equation they will also be linear in the solution of (24). It is hence convenient to separate the solution into two parts.

First we assume that the right side of (24) is

$$g(x) = -\sqrt{1-x^2}\sum_{n=1}^{\infty} A_{2n}\varepsilon^{2n}\frac{2n+1}{2n-1}\beta_n E_{n-1}(x^2)$$

We denote its corresponding solution by

$$p^\circ(x) = -4A_2\left[\sum_{n=0}^{\infty} p_n{}^\circ(x)\,\varepsilon^n + \sum_{n=1}^{\infty}\sum_{m=0}^{\infty} p^\circ{}_{m,n}(x)\,\varepsilon^{3n+m-1}\ln^n\varepsilon\right] \tag{36}$$

By the method set forth in [10] we obtain

$$p_0{}^\circ(x) = p_1{}^\circ(x) = p^\circ{}_{0,n}(x) = p^\circ{}_{1,n}(x) = p^\circ{}_{2,n}(x) = 0$$

$$p_2{}^\circ(x) = \sqrt{1-x^2},\quad p_3{}^\circ(x) = -\frac{\varkappa}{\theta_1\pi^2}J_3(x),\quad p^\circ{}_{3,1}(x) = \frac{c_2}{3\theta_1\pi^2}\sqrt{1-x^2}$$

$$p_4{}^\circ(x) = \frac{8}{9}\frac{A_4}{A_2}\sqrt{1-x^2}\,E_1(x^2) + \frac{c_1}{2\theta_1}\left[1-\frac{1}{9}E_1(x^2)\right]\sqrt{1-x^2} + $$
$$+\left(\frac{\varkappa}{\theta_1\pi^2}\right)^2\int_0^1 J_3(t)\,L(x,t)\,dt$$

$$p_5{}^\circ(x) = \frac{1}{3\theta_1\pi^2}\left\{c_3\sqrt{1-x^2} + c_2\left(J_2 - J_3 - \frac{1}{3}J_4\right) + \varkappa\left[2\sqrt{1-x^2} + 3J_3 - \right.\right.$$
$$\left.\left. - 5J_4 - 8\frac{A_4}{A_2}\left(J_3 - \frac{2}{3}J_4\right)\right]\right\} - \frac{\varkappa c_1}{(\theta_1\pi^2)^2}\left[\frac{\pi^2}{3}\left(J_3 + \frac{1}{3}J_4\right) + \right. \tag{37}$$
$$\left. + \int_0^1 L(x,s)\,ds\int_0^1 J_3(t)\frac{4t}{s+t}K\left(\frac{2\sqrt{st}}{s+t}\right)dt\right] - \left(\frac{\varkappa}{\theta_1\pi^2}\right)^3\int_0^1 dt\,L(x,t)\int_0^1 J_3(s)\,L(t,s)\,ds$$

Furthermore, we set the right side of the integral equation (24) as

$$g(x) = \int_0^1 \frac{B(x,t)}{\sqrt{1-t^2}}\,dt$$

We denote its corresponding solution by

$$p_0(x) = \sum_{n=0}^{\infty} q_n(x)\,\varepsilon^n + \sum_{n=1}^{\infty}\sum_{m=0}^{\infty} q_{m,n}(x)\,\varepsilon^{3n+m-1}\ln^n\varepsilon \tag{38}$$

Exactly as above we obtain

$$q_0(x) = -\varkappa J_1(x),\quad q_1(x) = c_1\pi^2\sqrt{1-x^2} + \frac{\varkappa^2}{\theta_1\pi^2}\int_0^1 J_1(t)\,L(x,t)\,dt$$

$$q_2(x) = (c_3 + 2\varkappa)\sqrt{1-x^2} + \varkappa\,[J_1(x) - 3J_3(x)] + c_2\,[J_2(x) - J_3(x)] -$$
$$-\varkappa\frac{c_1}{\theta_1}\left[J_3(x) + \frac{1}{\pi^2}\int_0^1 ds\,L(x,s)\int_0^1 J_1(t)\frac{4t}{s+t}K\left(\frac{2\sqrt{st}}{s+t}\right)dt\right] -$$
$$-\frac{\varkappa^3}{(\theta_1\pi^2)^2}\int_0^1 dt\,L(x,t)\int_0^1 J_1(s)\,L(t,s)\,ds$$

$$q_{0,1}(x) = c_2\sqrt{1-x^2}$$

Therefore, the solution of the integral equation (24) with the right side (25) is representable as

$$p(x) = ap^*(x) + p^\circ(x) + cp_0(x) \tag{39}$$

where the functions with the asterisk should be evaluated by means of (36), (37) by inserting therein

$$A_{2n} = (-1)^{n-1}\frac{1\cdot 3\cdot\ldots\cdot(2n-1)(1+4n)}{2^{n-1}n!}$$

Substituting the solution (39) into the conditions (26) and (28) and taking account of (27), we obtain a system of two linear algebraic equations in the constants c and a. The problem can be considered solved.

In order to obtain the solution of the contact problem in the class of bounded functions, we should set $c = 0$ in (23) and (39). The formula for the approach (28) is hence simplified and becomes

$$a\left\{2 + \frac{1}{\pi^2}\int_0^1 p^*(t)\left[2\pi + \frac{\varepsilon}{\theta_1} S(0, t)\right]dt\right\} = A_0 - \frac{1}{\pi^2}\int_0^1 p^\circ(t)\left[2\pi + \frac{\varepsilon}{\theta_1} S(0, t)\right]dt$$

Substituting in this relationship the corresponding series for p°, p^* and $S(0, t)$ we obtain

$$a = \frac{A_0}{2} + \left(\frac{5}{2} A_0 + A_2\right)\varepsilon^2\left\{1 + \frac{2\varepsilon}{3\theta_1\pi^2}\left[c_0 + \varkappa\left(\frac{5}{6} - \ln 4\varepsilon\right)\right]\right\} + O(\varepsilon^4) \qquad (40)$$

Up to now the unknown boundary of the domain of contact ($\varepsilon = \operatorname{tg} {}^1/{}_2\gamma$) is defined by the condition (26) in which we must set $c = 0$. Taking account of (39) and (40), we obtain

$$\frac{z}{(2\varepsilon)^3} = \frac{Z}{A_2 R^2}\,\frac{3\theta_1\pi}{8G\varepsilon^3} = \left(1 + \frac{5}{2}\frac{A_0}{A_2}\right)\left\{1 - \varepsilon\,\frac{3}{4}\,\frac{\varkappa}{\theta_1\pi^2} + \varepsilon^2\left[4 + \frac{8}{5}\frac{A_4}{A_2} + \right.\right.$$

$$\left. + \frac{2}{5}\frac{c_1}{\theta_1} + 3\left(\frac{\varkappa}{\theta_1\pi^2}\right)^2\int_0^1 t\sqrt{1 - t^2}\,J_3(t)\,dt\right] + \varepsilon^3 \ln\varepsilon\,\frac{c_2 - 10\varkappa}{3\theta_1\pi^2} + \qquad (41)$$

$$+ \frac{\varepsilon^3}{3\theta_1\pi^2}\left[10c_0 + c_3 + c_2\left(\ln 4 - \frac{7}{4}\right) + \varkappa\left(\frac{11}{12} - \frac{10}{3}\frac{A_4}{A_2} - 10\ln 4 - \frac{11}{6}\frac{c_1}{\theta_1}\right) - \right.$$

$$\left.\left. - \frac{9\varkappa^3}{(\theta_1\pi^2)^2}\int_0^1 sJ_2{}^2(s)\,ds\right]\right\} - \frac{20}{3}\frac{A_0}{A_2}\left(\frac{27}{20} + \frac{A_4}{A_2}\right)\varepsilon^2\left(\frac{3}{5} - \varepsilon\,\frac{5}{12}\,\frac{\varkappa}{\theta_1\pi^2}\right) + O(\varepsilon^4)$$

Limiting ourselves to the first terms in (36), (40) and (41), we obtain the solution of the problem of impression of a parabolic stamp into an elastic half-space.

It should be noted that the solution of the integral equation (24) constructed above can be obtained by successive approximations with subsequent grouping of terms having the same order of smallness in ε.

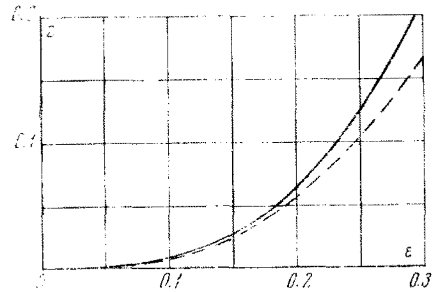

Fig. 2

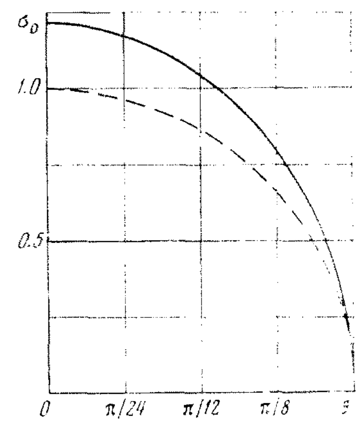

Fig. 3

As an illustration, let us examine the problem of the contact between a heavy elastic sphere $r \leqslant R$ of density ρ and a spherical stamp of similar radius $r = R(1 + \Delta)$. It is easy to find the particular solution of the elasticity theory equations corresponding to the mass forces

$$\mathbf{K} = \rho g \mathbf{e}_z$$

$$u_r = -r^2 \cos\theta \frac{1-2\nu}{1+\nu}\frac{\rho g}{4G}, \qquad u_\theta = -r^2 \cos\theta \frac{1-2\nu}{1+\nu}\frac{\rho g}{4G}$$

The corresponding stresses are

$$\sigma_r = \sigma_\theta = \sigma_\varphi = -r\cos\theta\,\rho g, \qquad \tau_{r\theta} = 0$$

It can be shown that

$$\int_0^\pi \cos\alpha\, H(\theta, \alpha)\sin\alpha\, d\alpha = 0, \qquad 0 < \theta < \pi$$

Therefore, the problem is reduced to solving the integral equation (9) with the right side

$$v(\theta) = -a\cos\theta + \sqrt{1 + 2\Delta + \Delta^2\cos\theta} - 1 - \Delta\cos\theta$$

$$a = a_0 - \frac{1-2\nu}{1+\nu}\frac{\rho g R}{4G}$$

where a_0 is the approach between the stamp and the center of the sphere.

The solution is given by (39)–(41) in which we should take

$$A_0 = 0, \qquad A_2 = \frac{4\Delta}{1+\Delta}, \qquad \frac{A_4}{A_2} = -\frac{1}{2}\frac{3 + 4\Delta - \Delta^2}{(1+\Delta)^2}$$

Presented in Fig. 2 is the dependence (41) for $\nu = 0.3$, $\Delta = 0.001$. Given in Fig. 3 is the contact pressure distribution when the angle of contact is $2\gamma = 60°$, the angle θ is laid off on the horizontal axis, while along the vertical the quantity

$$\sigma_0(\theta) = -\sigma(\theta) \Big/ \frac{2\varepsilon}{1+\varepsilon^2}\frac{A_2 G}{\pi(1-\gamma)}$$

is plotted.

The dashed curves in both figures refer to the Hertz solution. A comparison shows that for large domains of contact ($\gamma \geqslant 30°$) the Hertz solution, obtained under the assumption that the contacting bodies can be replaced by half-spaces, results in considerable errors ($\delta > 20\%$).

The author is grateful to L. S. Barkov for taking on the task of performing all the necessary computations.

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Translated by M. D. F.